

Analytical approximations to the core radius and energy of magnetic vortex in thin ferromagnetic disks.

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The energy of magnetic vortex core and its equilibrium radius in thin circular cylinder were first presented by N.A. Usov and S.E. Peschany in 1994. Yet, the magnetostatic function, entering the energy expression, is hard to evaluate and approximate. In this communication precise and explicit analytical approximations to this function (as well as equilibrium vortex core radius and energy) are derived in terms of elementary functions. Also, several simplifying approximations to the magnetic Hamiltonian and their impact on theoretical stability of magnetic vortex state are discussed.

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The first topological soliton was discovered as a solution of non-linear field theory equations by Anthony Skyrme.¹ It had a form of three-dimensional hedgehog and was named subsequently “skyrmion” in honor of the discoverer. After the landmark work of A.A. Belavin and A.M. Polyakov,² topological solitons have crossed the boundary into condensed matter physics. The latter authors discovered much more topological soliton solutions in the infinite Heisenberg ferromagnet, as many as there are rational functions of complex variable, mapping any such function into some equilibrium magnetic structure. Zeros of numerator (possibly with high multiplicity), of these rational functions, correspond to the centers of magnetic vortices, while zeros of denominator to the centers of magnetic anti-vortices. Thanks to the infinite size of the 2D ferromagnet, considered by Belavin and Polyakov, solitons appeared as absolutely stable. Once magnetic texture with a certain topological charge (number of magnetic vortices minus the number of magnetic anti-vortices, including the multiplicities of corresponding zeros into the count) have been created, all the structures with different topological charge appear to be separated by an infinite energy barrier.

The model of Belavin and Polyakov became known back in particle physics as non-linear $O(3)$ σ model in 3+1 dimensions and reformulated elegantly in terms of functions of complex variable by G. Woo.³ David J. Gross found additional family of “meron” solutions to it.⁴ Since then, the original Belavin-Polyakov solutions became known as just “solitons”. Merons (and all other $O(3)$ σ model solutions besides solitons³) have infinite energy in unbounded 2-d ferromagnet, but can be realized when the ferromagnet is finite.⁵

While these solutions were obtained long ago, the question of their stability has a history of its own. Kosterlitz and Thouless⁶ analyzed the stability of vortices in 2D ferromagnet and came to conclusion that they are unstable and such order could not exist. It is, indeed, true that the energy of Belavin-Polyakov solitons is scale-invariant and their size is, thus, undefined. In real ferromagnets, however, there are various other interactions (not exotic at all), which make the vortices stable. N.A. Usov and

S.E. Peschany, were the first to show that dipolar magnetostatic interaction fully stabilizes magnetic vortex in ferromagnetic cylinder both with respect to core radius change⁷ and vortex center displacement.⁸

Here, starting from more recent (and more general) description of magnetization distributions in finite nano-elements via functions of complex variable⁹ the impact of various approximations on vortex stability is reviewed in unified manner and the expression for vortex core radius in circular cylinder⁷ is re-derived. This expression defines the core radius implicitly via an equation and an integral of certain special functions, which is very inconvenient to evaluate and approximate at small cylinder thickness due to analyticity problems. It is, however, possible to introduce small parameters and expand the integrals involved and the vortex core radius into series, deriving an explicit analytical approximate (but very precise) expressions, presented at the end.

In finite planar nano-elements the equilibrium magnetization configurations can be described via rational functions of complex variable with real coefficients⁹ (as opposed to complex coefficients in the case of infinite film^{2,3}). The simplest ansatz for magnetic vortex in circular cylinder (of thickness L_Z and radius R) can be written in this language as

$$f(z) = \imath(z - a)/R_V \quad (1)$$

where $z = X + \imath Y$ with X and Y being the Cartesian coordinates in the cylinder's plane (the magnetization distribution is assumed to be independent on out-of-plane coordinate Z), R_V is the vortex core radius and a is the displacement of the vortex from the origin ($a = 0$ corresponds to the centered vortex). Let us then define a complex function

$$w(z, \bar{z}) = \begin{cases} f(z) & |f(z)| \leq 1 \\ f(z)/|f(z)| & |f(z)| > 1 \end{cases}, \quad (2)$$

where the line over variable denotes complex conjugation. The function w is shown to depend explicitly on both z and \bar{z} because it is, in general, not holomorphic. It consists of two parts: soliton (where it is analytic and

$\partial w(z, \bar{z})/\partial \bar{z} = 0$) and meron (where $w\bar{w} = 1$, joined at a line (possibly multiply-connected if there are several vortices or anti-vortices) $|f| = 1$). The magnetization components, normalized by material's saturation magnetization M_S , are then expressed via stereographic projection as

$$m_x + im_y = \frac{2w(z, \bar{z})}{1 + w(z, \bar{z})\bar{w}(z, \bar{z})} \quad (3)$$

$$m_z = \frac{1 - w(z, \bar{z})\bar{w}(z, \bar{z})}{1 + w(z, \bar{z})\bar{w}(z, \bar{z})}. \quad (4)$$

If written via magnetization components, this ansatz is exactly equivalent the one by Usov and Peschany⁷ and also belongs to the class of trial functions, considered by Kosterlitz and Thouless.⁶ Following the latter work, let us first take into account only the exchange interaction. In complex notation the exchange energy density (omitting the factor $C/2$, where C is the exchange stiffness) can be directly expressed via the function w :

$$\sum_{i=x,y,z} (\vec{\nabla} m_i)^2 = \frac{8}{(1 + w\bar{w})^2} \left(\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right), \quad (5)$$

where $\partial/\partial z = (\partial/\partial X - i\partial/\partial Y)/2$ and $\partial/\partial \bar{z} = (\partial/\partial X + i\partial/\partial Y)/2$. The total exchange energy can be obtained by integrating this density over nano-element's volume. Recalling the Riemann-Green theorem

$$\frac{1}{2i} \oint_{\partial \mathcal{D}} u(\zeta, \bar{\zeta}) d\zeta = \iint_{\mathcal{D}} \frac{\partial u(z, \bar{z})}{\partial \bar{z}} dX dY, \quad (6)$$

where u is a complex function of the complex argument (not necessary analytic¹⁰), it is possible to reduce the area integral over cylinder's face \mathcal{D} for the total exchange energy to a contour integral over its boundary $\partial \mathcal{D}$, provided there is a complex function, whose derivative over \bar{z} yields the exchange energy density (5). Luckily, such function (actually two functions, one for soliton and one for meron part of w) can be easily obtained by direct integration:

$$u^S(z, \bar{z}) = -\frac{8}{1 + f(z)\bar{f}(\bar{z})} \frac{1}{f(z)} \frac{\partial f}{\partial z}, \quad (7)$$

$$u^M(z, \bar{z}) = \frac{1}{f(z)} \frac{\partial f}{\partial z} \log(f(z)\bar{f}(\bar{z})). \quad (8)$$

Thus, from (6), the total exchange energy inside the soliton is

$$\frac{E_{\text{EX}}^S}{CL_Z/2} = \frac{2}{i} \oint_{|f(\zeta)|=1} \frac{1}{f(\zeta)} \frac{\partial f(\zeta)}{\partial \zeta} d\zeta, \quad (9)$$

where the fact that $|f(\zeta)| = 1$ on the integration contour was used and the additional minus sign appears because the original contour of integration had to be walked clockwise. The function under the integral is analytic everywhere except the vortex centers z_i , where $f(z_i) = 0$. Assuming that line $|f(\zeta)| = 1$ does not cross the particle boundary, it is possible to tighten the contours around

each topological singularity (vortex or anti-vortex center) and use the residue theorem

$$\frac{E_{\text{EX}}^S}{CL_Z/2} = 4\pi \sum_i \text{Res} \left. \frac{1}{f(z)} \frac{\partial f(z)}{\partial z} \right|_{z \rightarrow z_i}. \quad (10)$$

In particular, for $f(z)$ from Eq. 1 this gives $E^S/(CL_Z/2) = 4\pi$. For an arbitrary configuration of vortices and anti-vortices this number will be multiplied by their total number, including multiplicities.

For the meron part, the integration boundary is multiply-connected. However, on the inner boundaries (encircling solitons) $|f(\zeta)| = 1$ and $u^M(z, \bar{z}) \sim \log 1 = 0$. Thus, only the integral over the cylinder's outer boundary remains

$$\frac{E_{\text{EX}}^M}{CL_Z/2} = \frac{1}{2i} \oint_{\partial \mathcal{D}} \frac{1}{f(\zeta)} \frac{\partial f(\zeta)}{\partial \zeta} \log f(\zeta) \bar{f}(\bar{\zeta}) d\zeta \quad (11)$$

for $f(z)$ from Eq. 1 and the nano-element, shaped as circular cylinder ($\partial \mathcal{D}$ is $|z| = R$)

$$\begin{aligned} \frac{E_{\text{EX}}^M}{CL_Z/2} &= \int_0^{2\pi} \frac{(1 - a \cos(\varphi)/R) \log\left(\frac{a^2 - 2aR \cos(\varphi) + R^2}{R_V^2}\right)}{2(a^2/R^2 - 2a \cos(\varphi)/R + 1)} d\varphi \\ &= \pi \log\left(1 - \frac{a^2}{R^2}\right) - 2\pi \log\left(\frac{R_V}{R}\right), \end{aligned} \quad (12)$$

and the total exchange energy $e_{\text{EX}} = (E^S + E^M)/(\mu_0 \gamma_B M_S^2 \pi R^2 L_Z)$ (in subsequent text all the dimensionless energies, denoted by small letter e with different sub-/superscripts, are in this normalization) is

$$e_{\text{EX}} = \frac{L_E^2}{R^2} \left(2 - \log \frac{R_V}{R} + \log \sqrt{1 - \frac{a^2}{R^2}} \right), \quad (13)$$

where $\gamma_B = 4\pi$, $\mu_0 = 1$ in CGS units and $\gamma_B = 1$ in SI¹¹ and the exchange length $L_E = \sqrt{C/(\mu_0 \gamma_B M_S^2)}$ ¹². It can be seen immediately that this energy decreases with increasing the vortex core size R_V . The above expression is, formally, valid only for $R_V < R$, but it can be easily shown that the energy continues to decrease for larger R_V , reaching equilibrium for $R_V \rightarrow \infty$. This confirms the conclusion of Kosterlitz and Thouless⁶ that magnetic vortices are unstable when only the exchange interaction is taken into account.

There is, however, another interaction, present in all magnets. Namely, the long-range dipolar interaction between the local magnetic moments. Strictly speaking, this interaction is not instantaneous and its speed is limited by the speed of light. The account for retardation effects, however, contributes to the dissipation.¹³ It is convenient (if magnetic nano-elements are small enough and characteristic timescales are large enough) to consider the dipolar interaction in magnetostatic approximation, making it non-local. This non-locality still poses a major mathematical difficulty, since it makes the equations for equilibrium magnetization distribution not only non-linear partial differential, but also integral. To alleviate

this difficulty a number of local magnetostatic approximations had been developed. The most common (and very useful for considering domain walls in thin films) is based on using the local uniaxial in-plane anisotropy term $K_{\text{MS}}m_z^2$ instead of magnetostatic interaction. Selecting the local bulk anisotropy $K_{\text{MS}} = \mu_0\gamma_B M_S^2/2$, one gets the exact correspondence between the approximate and exact magnetostatic energy density in two limiting cases: when the film is magnetized in-plane (in this case magnetostatic energy is 0) and out-of-plane (in which case the energy is $\mu_0\gamma_B M_S^2/2$). Nano-elements have additional side surfaces and it was recently proposed by Kohn and Slustikov¹⁴ to use a similar expression for local surface anisotropy of magnetostatic origin on all surfaces with some a priori unknown constant K_s , replacing the m_z by a normal magnetization component on the surface. Let us try this approach.

When vortex is completely inside the particle and is centered ($a = 0$) the meron does not contribute to the anisotropy energy, and the contribution of soliton part is

$$e_A^{\text{in}} = \frac{1}{R^2} \int_0^{R_V} \frac{(r^2 - R_V^2)^2}{(r^2 + R_V^2)^2} r dr = \frac{R_V^2}{R^2} \frac{(3 - 4 \log 2)}{2}. \quad (14)$$

The total energy density $e_{\text{EX}} + e_A^{\text{in}}$ at $a = 0$ now has a minimum when $R_V^A = L_E/\sqrt{3 - 4 \log 2}$. Or, approximately, $R_V^A \approx 2.09698 L_E = 0.59155 \sqrt{4\pi} L_E$, independent on cylinder's thickness L_Z . This thickness independence is the result of expressing the magnetostatic energy in the form of surface anisotropy, and, as will be seen later, is wrong. Nevertheless, unlike the purely exchange approximation, the vortex core size is now stable. It is also worth noting that this approximation is exact in the limit of vanishing film thickness, so that $R_V^A/L_E = \lim_{\lambda \rightarrow 0} \rho_V(\lambda)$, where $\rho_V(\lambda)$ is the vortex core radius, computed with full treatment of magnetostatics (23).

But stable vortex core size is not all, the vortex must also be stable with respect to the displacement of its center. To consider this, accounting of the magnetostatically-defined anisotropy on the cylinder's face is not enough, since (for the case of vortex inside the particle) its energy is independent on the vortex center displacement a . But the exchange energy (13) decreases when vortex is displaced ($|a|$ increases from 0) and this leads to instability. To consider this case properly within the magnetostatic anisotropy approximation let us follow the proposal of Kohn and Slustikov¹⁴ and introduce additional surface anisotropy K_S on the cylinder's side. It gives the following contribution to the energy of displaced vortex, assuming it is fully within the particle

$$E_A^s = K_S L_Z R \int_0^{2\pi} \left(\Re \frac{e^{-i\varphi} f(Re^{i\varphi})}{|f(Re^{i\varphi})|} \right)^2 d\varphi = \frac{K_S L_Z \pi a^2}{R}. \quad (15)$$

The exchange energy (12) can be expanded as $E_{\text{EX}}(a) \approx \text{const} - CL_Z \pi a^2 / (2R^2)$. Equating two second order terms

in a gives the condition for vortex stability with respect to displacement: $R > R_S^A = C/(2K_S)$. For radii, smaller than R_S^A , the vortex is unstable. This is, again, only partially correct. The stability condition turns out to be independent on L_Z , which means that, while the particles of very small radii are correctly single-domain, in particles of disappearing thickness the vortex state is unconditionally stable, which is qualitatively wrong. Nevertheless, if one deals with particles of specific size and considers K and K_S as free parameters, the approach of Kohn and Slustikov¹⁴ may yield a reasonable approximation to the stability and evolution of vortex state, in this case K_S will have to vanish as $L_Z \rightarrow 0$. The advantage of this approach is simplicity, as it allows to get explicit expressions for most interesting quantities. The full account for long-range magnetostatic interaction, which is necessary to build the vortex state theory without free parameters, is much harder to do. Yet, in the following text, approximate expressions for vortex radius and energy with full account of magnetostatics are derived, which are almost as simple.

To compute the magnetostatic energy let us use the magnetic charges formalism, introducing a magnetic charge density $-\text{div } \vec{m}$ which is automatically equal to the normal component of magnetization on the surface of magnetic material (in which case it is a surface charge density σ). In centered vortex $a = 0$ there is only a face charge (surface charge on cylinder's face, proportional to m_z), equal to

$$\sigma(r) = M_S \frac{R_V^2 - r^2}{R_V^2 + r^2}, \quad (16)$$

where $r < R_V$ because all the charge is concentrated in the vortex core. The interaction energy of two such systems of charge at parallel planes (faces of the cylinder), separated by distance h , can be directly written as

$$\frac{4\pi U(h)}{\mu_0 \gamma_B} = \int_0^{R_V} \int_0^{2\pi} \int_0^{R_V} \int_0^{2\pi} \frac{\sigma(r_1) \sigma(r_2) r_1 dr_1 d\varphi_1 r_2 dr_2 d\varphi_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_1 - \varphi_2) + h^2}}. \quad (17)$$

It is possible to obtain two equivalent representations for this integral, one by directly integrating over the angles

$$\int_0^{2\pi} \int_0^{2\pi} \frac{d\varphi_1 d\varphi_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_1 - \varphi_2) + h^2}} = \frac{8\pi K\left(\frac{4r_1 r_2}{h^2 + (r_1 + r_2)^2}\right)}{\sqrt{h^2 + (r_1 + r_2)^2}}, \quad (18)$$

where $K(k)$ is a complete elliptic integral of the first kind, which gives

$$u(\chi) = \frac{2}{\pi} \int_0^1 \int_0^1 \frac{\rho_1(1 - \rho_1^2) \rho_2(1 - \rho_2^2) K\left(\frac{4\rho_1 \rho_2}{h^2 + (\rho_1 + \rho_2)^2}\right)}{(1 + \rho_1^2)(1 + \rho_2^2) \sqrt{\chi^2 + (\rho_1 + \rho_2)^2}}, \quad (19)$$

where the dimensionless quantities $u = U/(\mu_0\gamma_B M_S^2 \pi R_V^3)$, $\chi = h/R_v$, $\rho_1 = r_1/R_V$, $\rho_2 = r_2/R_V$ have been introduced. Another representation can be obtained using summation theorem for Bessel's functions of the first kind¹⁵

$$u(\chi) = \int_0^\infty e^{-k\chi} \left(\int_0^1 \frac{1-\rho^2}{1+\rho^2} J_0(k\rho) \rho d\rho \right)^2 dk, \quad (20)$$

which is shorter on paper, but, unlike (19), is, actually, a triple integral. The magnetostatic energy of the vortex core per unit of cylinder's volume is then

$$e_{\text{MS}} = \frac{E_{\text{MS}}}{\mu_0\gamma_B M_S^2 \pi L_Z R^2} = \frac{R_V^3}{L_Z R^2} (u(0) - u(L_Z/R_V)), \quad (21)$$

where the first term accounts for the face charge's self energy, while the second for interaction of charges on the opposite faces. The total dimensionless energy of the cylinder with centered vortex is

$$e = \frac{L_E^2}{R^2} \left(2 - \log \frac{R_V}{R} \right) + e_{\text{MS}}. \quad (22)$$

Minimization of this energy with respect to R_V results in the following equation

$$-\frac{1}{\rho_V} + \frac{3\rho_V^2(u(0) - u(\lambda/\rho_V))}{\lambda} + \rho_V u'(\lambda/\rho_V) = 0, \quad (23)$$

where $\rho_V = R_V/L_E$, $\lambda = L_Z/L_E$ and prime means derivative. This equation is independent on particle radius.

It is, of course, possible to evaluate the integrals (19), (20) on computer (but even this is tricky, since the second is a badly converging oscillating improper integral and the first contains a peak at $\rho_1 = \rho_2$, turning into a line of integrable logarithmic singularities when $h = 0$) and solve the transcendental equation (23) numerically, but this is far less convenient (and useful), compared to having their simple analytical expressions. Let us now obtain such expressions approximately.

The simplest is the case of large cylinder thickness, corresponding to $\chi \gg 1$. In this case the outer integral in (20) is converging very fast, and also the integrand in (19) is well behaved. This allows to perform straightforward Taylor's expansion of the integrand and perform the integration term by term, which gives:

$$u(\chi) = \frac{(\log(4) - 1)^2}{4\chi} + \frac{3 + 8\log^2(2) - 10\log(2)}{8\chi^3} + \frac{35 + 4\log(2)(18\log(2) - 25)}{32\chi^5} + \dots \quad (24)$$

Solving (23) with this u results in the following expansion for the equilibrium vortex core radius

$$\rho_V^{\text{EQ}}(\lambda) = \left(\frac{\lambda}{3u_0} \right)^{1/3} + \left(\frac{(\log(4) - 1)^6}{81^2 u_0^5 \lambda} \right)^{1/3} + \frac{(\log(4) - 1)^4}{81 \lambda u_0^3} + \dots, \quad (25)$$

where $u_0 = u(0) = 0.0826762$. Using this expansion, the equilibrium energy of thick cylinder ($\lambda \gg 1$) with magnetic vortex can be written as

$$e^{\text{EQ}} \approx \frac{7 + \log\left(\frac{3\rho^3 u_0}{\lambda}\right)}{3\rho^2} - \frac{(\log(4) - 1)^2}{12\rho^2 (3u_0^4 \lambda^2)^{1/3}}, \quad (26)$$

where $\rho = R/L_E$.

These expressions are simple and for $\lambda > 2$ are precise to a few percent (and for $\lambda/\sqrt{4\pi} > 1$ the precision is better than 1%). The problem, however, is that assumption of uniformity of magnetic texture along Z axis is not a good approximation for thick cylinders ($\lambda \gg 1$), which, eventually, start to develop a 3D structure (such as variation of vortex radius with Z at first). In other words, this approximation is precise mostly in the region, where the vortex (1) is far from the ground state of the system (Eq. 26 may still be useful to find the extent of this region by comparing it to the energy of other magnetization textures).

It might be tempting to expand the magnetostatic function $u(\chi)$ around $\chi = 0$ and build the approximate vortex state theory on top of that. The difficulty is that $u(\chi)$ is not analytic at this point (it has terms, proportional to $\chi \log \chi$) and the integrals for higher order Taylor expansion terms do not converge. Since the very thin particles are single-domain and also thin cylinders with large radius start to develop a domain structure or several vortices, bound as finite fragments of cross-tie domain walls, such approximation would also be the most precise in the region, where the vortex is not the ground state.

This suggests the idea to build the magnetostatic function expansion around an intermediate point $\chi = 1$, where the function $u(\chi)$ is analytic and otherwise well-behaved. Such expansion is most precise for $R_V \sim L_Z$, where all the physical assumptions of the vortex state theory are valid. This region is also close to the triple point on the magnetic phase diagram.¹⁶ The point $\chi = 1$ corresponds to

$$\lambda_0 = \frac{1}{\sqrt{3(u(0) - u(1)) + u'(1)}} \approx 2.7284 \approx 0.76967\sqrt{4\pi}. \quad (27)$$

The expansions for equilibrium vortex radius and energy about the point λ_0 are the following

$$\rho_V^{\text{EQ}}(\lambda) = \sum_{i=0}^{\infty} a_i (\lambda - \lambda_0)^i \quad (28)$$

$$e^{\text{EQ}} = \frac{\log(\rho/\lambda_0)}{\rho^2} + \frac{1}{\rho^2} \sum_{i=0}^{\infty} b_i (\lambda - \lambda_0)^i, \quad (29)$$

where the first few coefficients are

i	0	1	2	3	4
a_i	λ_0	0.189400	-0.012521	0.001182	-0.000093
b_i	2.387556	-0.082425	0.010332	-0.00171	0.000315

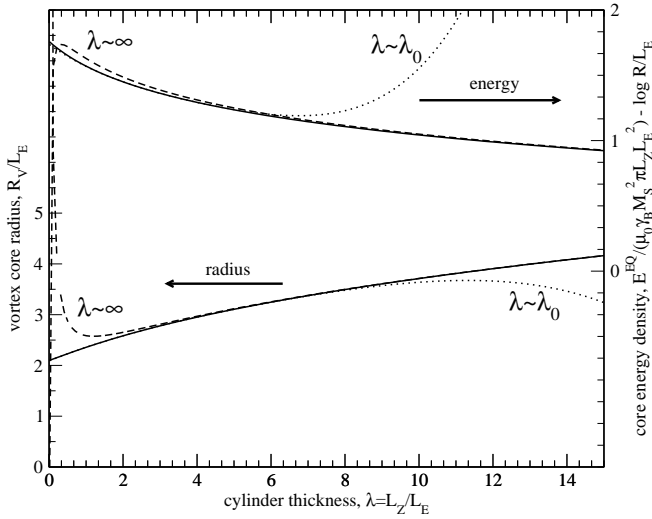


FIG. 1. Vortex core radius and energy density, as function of the cylinder thickness. The dotted and dashed lines show approximate analytical expressions, the solid lines are the result of the exact numerical calculation.

These coefficients are just universal dimensionless constants, they decay rapidly and are sufficient to obtain the values, precise up to a 0.5% in the interval $0 < \lambda < 6$. Comparison of the above analytical approximations with exact numerical values of core radius and energy are shown in Figure.

CONCLUSIONS

Starting with the description of magnetization distributions via complex variable, various simplifying physical approximations for magnetic Hamiltonian were consistently (in the same notations and units) presented and compared with their advantages and deficiencies highlighted. The formulas for the exchange energy of such distributions (10)-(11), provided the vortices and anti-vortices are fully contained inside the particle (which covers many distributions of Ref. 9), were presented here for the first time. Two simple and explicit analytical approximations for equilibrium vortex core radius and energy in circular cylinder were derived, which, together, cover the whole range of cylinder geometries.

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- ¹ T. Skyrme, Nuclear Physics A. **31**, 556 (1962)
- ² A. A. Belavin and A. M. Polyakov, ZETP lett. **22**, 245 (1975), (in Russian)
- ³ G. Woo, Journal of Mathematical Physics **18**, 1264 (1977)
- ⁴ D. J. Gross, Nuclear Physics B **132**, 439 (1978)
- ⁵ K. L. Metlov, "Two-dimensional topological solitons in small exchange-dominated cylindrical ferromagnetic particles," (2000), [arXiv:cond-mat/0012146](#)
- ⁶ J. M. Kosterlitz and D. J. Thouless, Journal of Physics C: Solid State Physics **6**, 1181 (1973)
- ⁷ N. A. Usov and S. E. Peschany, J. Magn. Magn. Mater. **118**, L290 (1993)
- ⁸ N. A. Usov and S. E. Peschany, Fiz. Met. Metal (in Russian) **12**, 13 (1994)
- ⁹ K. L. Metlov, Phys. Rev. Lett. **105**, 107201 (2010)
- ¹⁰ For analytic u the double integral over \mathcal{D} is equal to 0, which is the manifestation of Cauchy theorem.
- ¹¹ A. Aharoni, *Introduction to the theory of ferromagnetism* (Oxford University Press, Oxford, 1996) ISBN 0198517912

- ¹² The other common definition of the exchange length $L_E^{UP} = \sqrt{C/(\mu_0 M_S^2)}$, used by Usov and Peschany⁷ and in many followup works, is, actually, dependent on system of measurement units and makes the formulas for the dimensionless energy and all the derived quantities depend on units too. To avoid this complication the definition $L_E = \sqrt{C/(\mu_0 \gamma_B M_S^2)}$ is adopted here, which in CGS units (which are almost exclusively used in conjunction with L_E^{UP} definition) is by factor $\sqrt{4\pi}$ smaller (making all the lengths, measured in units of L_E , by factor $\sqrt{4\pi}$ larger than the lengths, measured in L_E^{UP}). The numeric quantities here are given for both definitions of the exchange length to make the comparison to other results in the literature easier.
- ¹³ T. Bose and S. Trimper, Phys. Rev. B **83**, 134434 (2011)
- ¹⁴ R. V. Kohn and V. V. Slastikov, Archive for Rational Mechanics and Analysis **178**, 227 (2005)
- ¹⁵ K. Y. Guslienko and K. L. Metlov, Phys. Rev. B **63**, 100403R (2001)
- ¹⁶ K. L. Metlov and K. Y. Guslienko, J. Magn. Magn. Mater. **242-245**, 1015 (2002)